

LATTICE-POINT GENERATING FUNCTIONS FOR FREE SUMS OF CONVEX SETS

MATTHIAS BECK, PALLAVI JAYAWANT, AND TYRRELL B. MCALLISTER

ABSTRACT. Let \mathcal{J} and \mathcal{K} be convex sets in \mathbb{R}^n whose affine spans intersect at a single rational point in $\mathcal{J} \cap \mathcal{K}$, and let $\mathcal{J} \oplus \mathcal{K} = \text{conv}(\mathcal{J} \cup \mathcal{K})$. We give expressions for the generating function

$$\sigma_{\mathcal{J} \oplus \mathcal{K}}(z_1, \dots, z_n) = \sum_{(m_1, \dots, m_n) \in (\mathcal{J} \oplus \mathcal{K}) \cap \mathbb{Z}^n} z_1^{m_1} \dots z_n^{m_n}$$

in terms of $\sigma_{\mathcal{J}}$ and $\sigma_{\mathcal{K}}$, under certain conditions on \mathcal{J} and \mathcal{K} . This work is motivated by (and recovers) a product formula of B. Braun for the Ehrhart series of $\mathcal{P} \oplus \mathcal{Q}$ in the case where \mathcal{P} and \mathcal{Q} are lattice polytopes, one of which is reflexive.

1. INTRODUCTION

The convex subsets of \mathbb{R}^n constitute a distributive lattice in which the join $\mathcal{J} \oplus \mathcal{K}$ of $\mathcal{J}, \mathcal{K} \subseteq \mathbb{R}^n$ is the convex hull of their union: $\mathcal{J} \oplus \mathcal{K} := \text{conv}(\mathcal{J} \cup \mathcal{K})$.¹ We call the join $\mathcal{J} \oplus \mathcal{K}$ a *free sum of \mathcal{J} and \mathcal{K}* when \mathcal{J} and \mathcal{K} each contain the origin and their respective linear spans are orthogonal coordinate subspaces (*i.e.*, subspaces spanned by subsets of the standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$).² More generally, we will write “ $\mathcal{J} \oplus \mathcal{K}$ is a free sum” when $\mathcal{J} \oplus \mathcal{K}$ is a free sum of \mathcal{J} and \mathcal{K} up to the action of $\text{SL}_n(\mathbb{Z})$ on \mathbb{R}^n . A familiar example is the octahedron $\text{conv}\{\pm \mathbf{e}_1, \pm \mathbf{e}_2, \pm \mathbf{e}_3\}$ in \mathbb{R}^3 , which is the free sum of the “diamond” $\text{conv}\{\pm \mathbf{e}_1, \pm \mathbf{e}_2\}$ and the line segment $\text{conv}\{\pm \mathbf{e}_3\}$.

Our goal is to understand the integer lattice points in a free sum and its integer dilates in terms of the corresponding data for its summands. Of particular interest is the case of a free sum $\mathcal{P} \oplus \mathcal{Q}$ in which \mathcal{P} and \mathcal{Q} are rational polytopes. A *rational* (respectively, *lattice*) *polytope* in \mathbb{R}^n is a polytope all of whose vertices are in \mathbb{Q}^n (respectively, the *integer lattice* \mathbb{Z}^n). Given a rational polytope $\mathcal{P} \subseteq \mathbb{R}^n$, its *Ehrhart series*

$$\text{Ehr}_{\mathcal{P}}(t) := 1 + \sum_{k \in \mathbb{Z}_{\geq 1}} |k\mathcal{P} \cap \mathbb{Z}^n| t^k$$

is the generating function of the *Ehrhart quasi-polynomial* of \mathcal{P} , which counts the integer lattice points in $k\mathcal{P}$ as a function of an integer dilation parameter k . Let $\text{den } \mathcal{P}$ denote the *denominator* of \mathcal{P} , the smallest positive integer such that the

Date: 30 June 2012.

¹See [4] for a study of lattices of convex sets in \mathbb{R}^n .

²The free sum is sometimes called the *direct sum*. Diverse conditions on the summands appear in the literature. Some authors require that the origin [2], or at least a unique point of intersection [12, 14], be in the interior of each summand. Others require no intersection, insisting only that the linear spans of the summands be orthogonal coordinate subspaces [5, 9]. We require each summand to contain the origin, but we allow the origin to be on the boundary.

corresponding dilate of \mathcal{P} is a lattice polytope. A famous theorem of Ehrhart [7] says that

$$\text{Ehr}_{\mathcal{P}}(t) = \frac{\delta_{\mathcal{P}}(t)}{(1 - t^{\dim \mathcal{P}})^{\dim \mathcal{P} + 1}}$$

for some polynomial $\delta_{\mathcal{P}}(t)$, the δ -polynomial of \mathcal{P} . (Common alternative names for the δ -polynomial include h^* -polynomial and Ehrhart h -vector). See, e.g., [3, 10, 15] for this and many more facts about Ehrhart series.

Our work is motivated by the following result of B. Braun, which expresses the δ -polynomial of $\mathcal{P} \oplus \mathcal{Q}$ in terms of the δ -polynomials of \mathcal{P} and \mathcal{Q} when \mathcal{P} is a reflexive polytope (defined in Section 3 below).

Theorem 1.1 ([5]). *Suppose that $\mathcal{P}, \mathcal{Q} \subseteq \mathbb{R}^n$ are lattice polytopes such that \mathcal{P} is reflexive, \mathcal{Q} contains the origin in its relative interior, and $\mathcal{P} \oplus \mathcal{Q}$ is a free sum. Then*

$$(1) \quad \delta_{\mathcal{P} \oplus \mathcal{Q}}(t) = \delta_{\mathcal{P}}(t) \delta_{\mathcal{Q}}(t).$$

In terms of Ehrhart series, Braun's equation (1) says that

$$\text{Ehr}_{\mathcal{P} \oplus \mathcal{Q}}(t) = (1 - t) \text{Ehr}_{\mathcal{P}}(t) \text{Ehr}_{\mathcal{Q}}(t)$$

when \mathcal{P} is reflexive and \mathcal{Q} contains the origin in its interior. Our first main result, Theorem 1.2 below, gives a multivariate generalization of equation (1) that weakens the condition that \mathcal{P} be reflexive. Before stating our multivariate generalization of Theorem 1.1, we first need to define some notation.

The Ehrhart series is a specialization of a multivariate Laurent series defined as follows. Let $\alpha: \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ be the affine embedding $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, 1)$. Given a convex set $\mathcal{K} \subseteq \mathbb{R}^n$, let $\text{cone } \mathcal{K} \subseteq \mathbb{R}^{n+1}$ be the set of all nonnegative scalar multiples of elements of $\alpha(\mathcal{K})$. Equivalently, $\text{cone } \mathcal{K}$ is the intersection of all linear cones containing $\alpha(\mathcal{K})$. Write $S_{\mathbb{Z}}$ for the set of integer lattice points in a set S . The *lattice-point generating function* $\sigma_S(\mathbf{z})$ of $S \subseteq \mathbb{R}^{n+1}$ is the formal multivariate Laurent series

$$\sigma_S(\mathbf{z}) := \sum_{\mathbf{m} \in S_{\mathbb{Z}}} \mathbf{z}^{\mathbf{m}}.$$

(Here we follow the convention of writing $\sigma_S(\mathbf{z})$ for $\sigma_S(z_1, \dots, z_{n+1})$ and $\mathbf{z}^{\mathbf{m}}$ for $z_1^{m_1} \dots z_{n+1}^{m_{n+1}}$, where $\mathbf{m} = (m_1, \dots, m_{n+1})$.) The Ehrhart series $\text{Ehr}_{\mathcal{P}}(t)$ then arises as a specialization of $\sigma_{\text{cone } \mathcal{P}}(\mathbf{z})$:

$$\text{Ehr}_{\mathcal{P}}(t) = \sigma_{\text{cone } \mathcal{P}}(1, \dots, 1, t).$$

Let $\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_{n+1}$ denote the standard basis vectors for \mathbb{R}^{n+1} . Given a closed cone $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ not containing $-\mathbf{e}_{n+1}$, define the projection $\varepsilon_{\mathcal{C}}: \mathcal{C} \rightarrow \partial \mathcal{C}$ (where “ ∂ ” denotes relative boundary) by letting

$$\varepsilon_{\mathcal{C}}(\mathbf{x}) := \mathbf{x} - \max \{ \lambda \in \mathbb{R} : \mathbf{x} - \lambda \mathbf{e}_{n+1} \in \mathcal{C} \} \mathbf{e}_{n+1}.$$

Given a compact convex set $\mathcal{J} \subseteq \mathbb{R}^n$, we write $\varepsilon_{\mathcal{J}}$ as an abbreviation for $\varepsilon_{\text{cone } \mathcal{J}}$. The *lower envelope* of \mathcal{C} is

$$\underline{\partial} \mathcal{C} := \varepsilon_{\mathcal{C}}(\mathcal{C}).$$

Thus, the lower envelope of \mathcal{C} is the set of points that are “vertically minimal” within \mathcal{C} . The *lower lattice envelope* of \mathcal{C} is

$$\underline{\partial}_{\mathbb{Z}} \mathcal{C} := \varepsilon_{\mathcal{C}}(\mathcal{C}_{\mathbb{Z}}).$$

Thus, the lower lattice envelope is the vertical projection of the lattice points in \mathcal{C} onto the lower envelope of \mathcal{C} . Observe that the lower lattice envelope is *not* necessarily the set $(\partial\mathcal{C})_{\mathbb{Z}}$ of lattice points in the lower envelope of \mathcal{C} . In general, some elements of $\partial_{\mathbb{Z}}\mathcal{C}$ may not be lattice points.

Theorem 1.2 (proved on p. 6). *Suppose that $\mathcal{J}, \mathcal{K} \subseteq \mathbb{R}^n$ are convex sets such that \mathcal{J} is compact and $\mathcal{J} \oplus \mathcal{K}$ is a free sum. Suppose moreover that $\partial_{\mathbb{Z}} \text{cone } \mathcal{J} = (\partial \text{cone } \mathcal{J})_{\mathbb{Z}}$. Then*

$$(2) \quad \sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})}(\mathbf{z}) = (1 - z_{n+1}) \sigma_{\text{cone } \mathcal{J}}(\mathbf{z}) \sigma_{\text{cone } \mathcal{K}}(\mathbf{z}).$$

We call equation (2) the *multivariate Braun equation*. Our second main result states that, when \mathcal{J} and \mathcal{K} are rational polytopes, the converse of Theorem 1.2 also holds. Given a rational polytope \mathcal{P} containing the origin, we observe (Proposition 3.2 below) that $\partial_{\mathbb{Z}} \text{cone } \mathcal{P} = (\partial \text{cone } \mathcal{P})_{\mathbb{Z}}$ if and only if the polar dual \mathcal{P}^{\vee} of \mathcal{P} (relative to its linear span) is a lattice polyhedron. We show that, if a free sum of rational polytopes satisfies the multivariate Braun equation, then the dual of one of those polytopes is a lattice polyhedron.

Theorem 1.3 (proved on p. 10). *Let $\mathcal{P}, \mathcal{Q} \subseteq \mathbb{R}^n$ be rational polytopes such that $\mathcal{P} \oplus \mathcal{Q}$ is a free sum. Then*

$$(3) \quad \sigma_{\text{cone } \mathcal{P} \oplus \mathcal{Q}}(\mathbf{z}) = (1 - z_{n+1}) \sigma_{\text{cone } \mathcal{P}}(\mathbf{z}) \sigma_{\text{cone } \mathcal{Q}}(\mathbf{z})$$

if and only if either \mathcal{P}^{\vee} or \mathcal{Q}^{\vee} is a lattice polyhedron.

After laying the groundwork for our approach to free sums in Section 2, we prove Theorem 1.2 and various corollaries, including Theorem 1.1, in Section 3. In Section 4, we give an expression for $\sigma_{\text{cone}(\mathcal{P} \oplus \mathcal{K})}$ when $\mathcal{P} \oplus \mathcal{K}$ is an arbitrary free sum in which \mathcal{P} is a rational polytope. We then use this expression to prove Theorem 1.3. Although Sections 3 and 4 address only those joins $\mathcal{J} \oplus \mathcal{K}$ that are free sums, our approach is not confined to this case. Section 5 studies the lattice-point generating function of $\text{cone}(\mathcal{J} \oplus \mathcal{K})$ when \mathcal{J} and \mathcal{K} intersect at an arbitrary unique rational point, under certain conditions on \mathcal{J} and \mathcal{K} . One case of interest that satisfies these conditions is a join $\mathcal{P} \oplus \mathcal{K}$ where \mathcal{P} is a Gorenstein polytope of index k intersecting an orthogonal convex set \mathcal{K} at the unique point $\mathbf{p} \in \mathcal{P}$ such that $k\mathbf{p}$ is a lattice point in the relative interior of $k\mathcal{P}$ (Corollary 5.8).

2. DECOMPOSITIONS OF CONES OVER FREE SUMS

We begin our study of the generating function $\sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})}$ from the vantage point of the following easy identity: Given any convex sets $\mathcal{J}, \mathcal{K} \subseteq \mathbb{R}^n$, their join $\mathcal{J} \oplus \mathcal{K}$ satisfies

$$(4) \quad \text{cone}(\mathcal{J} \oplus \mathcal{K}) = \text{cone } \mathcal{J} + \text{cone } \mathcal{K},$$

where the sum on the right is the Minkowski sum $S + T := \{s + t : s \in S, t \in T\}$. The goal of this section is to provide a series of refinements to equation (4). In general, this equation “double counts” elements of $\text{cone}(\mathcal{J} \oplus \mathcal{K})$, in the sense that there are many ways to express an element of the left-hand side as a sum from the right-hand side. Proposition 2.1 below gives a non-double-counting expression for $\text{cone}(\mathcal{J} \oplus \mathcal{K})$ in the spirit of equation (4) in the case where \mathcal{J} is compact, \mathcal{K} contains the origin, and their linear spans intersect trivially. Proposition 2.2 below provides a similar expression for the set of lattice points in $\text{cone}(\mathcal{J} \oplus \mathcal{K})$ when $\mathcal{J} \oplus \mathcal{K}$ is a free sum.

First we make a few additional notational remarks: We write $\pi: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ for the orthogonal projection

$$\pi: (x_1, \dots, x_n, x_{n+1}) \mapsto (x_1, \dots, x_n).$$

Given a subset S of \mathbb{R}^n or \mathbb{R}^{n+1} , let $\text{lin } S$ be the linear span of S . We say that two sublattices $\mathcal{L}, \mathcal{M} \subseteq \mathbb{Z}^n$ are *complementary sublattices of \mathbb{Z}^n* if each integer lattice point in $\text{lin}(\mathcal{L} \cup \mathcal{M})$ can be written uniquely as a sum of lattice points in \mathcal{L} and \mathcal{M} . Hence, when $\mathcal{J} \oplus \mathcal{K}$ is a free sum, $(\text{lin } \mathcal{J})_{\mathbb{Z}}$ and $(\text{lin } \mathcal{K})_{\mathbb{Z}}$ are complementary sublattices of \mathbb{Z}^n .

Equation (4) says that

$$\text{cone}(\mathcal{J} \oplus \mathcal{K}) = \bigcup_{\mathbf{x} \in \text{cone } \mathcal{J}} (\mathbf{x} + \text{cone } \mathcal{K}).$$

Using the concept of the lower envelope (defined in Section 1), we can make this union disjoint. This yields the desired “non-double-counting” version of equation (4). We use \bigsqcup to denote disjoint union.

Proposition 2.1. *Suppose that $\mathcal{J}, \mathcal{K} \subseteq \mathbb{R}^n$ are convex sets with \mathcal{J} compact and $\mathbf{0} \in \mathcal{K}$. Suppose in addition that the linear spans of \mathcal{J} and \mathcal{K} intersect trivially. Then*

$$\text{cone}(\mathcal{J} \oplus \mathcal{K}) = \bigsqcup_{\mathbf{x} \in \underline{\text{cone}} \mathcal{J}} (\mathbf{x} + \text{cone } \mathcal{K}).$$

Proof. We first show that the union on the right-hand side is a disjoint union. Suppose that

$$\mathbf{x}_1 + \mathbf{y}_1 = \mathbf{x}_2 + \mathbf{y}_2$$

for some $\mathbf{x}_1, \mathbf{x}_2 \in \underline{\text{cone}} \mathcal{J}$ and $\mathbf{y}_1, \mathbf{y}_2 \in \text{cone } \mathcal{K}$. Then we have $\pi(\mathbf{x}_1) + \pi(\mathbf{y}_1) = \pi(\mathbf{x}_2) + \pi(\mathbf{y}_2)$. Hence

$$\pi(\mathbf{x}_1) - \pi(\mathbf{x}_2) = \pi(\mathbf{y}_2) - \pi(\mathbf{y}_1) \in \text{lin } \mathcal{J} \cap \text{lin } \mathcal{K}$$

because the left-hand side of the equality is in $\text{lin } \mathcal{J}$ while the right-hand side is in $\text{lin } \mathcal{K}$. Since $\text{lin } \mathcal{J} \cap \text{lin } \mathcal{K} = \{\mathbf{0}\}$, it follows that $\pi(\mathbf{x}_1) = \pi(\mathbf{x}_2)$. Now, the preimage $\pi^{-1}(\pi(\mathbf{x}_1))$ contains exactly one point in $\underline{\text{cone}} \mathcal{J}$, so $\mathbf{x}_1 = \mathbf{x}_2$, proving disjointness.

It remains only to show that

$$\bigsqcup_{\mathbf{x} \in \underline{\text{cone}} \mathcal{J}} (\mathbf{x} + \text{cone } \mathcal{K}) = \bigcup_{\mathbf{x} \in \text{cone } \mathcal{J}} (\mathbf{x} + \text{cone } \mathcal{K}).$$

The left-hand side is contained in the right-hand side because $\underline{\text{cone}} \mathcal{J} \subseteq \text{cone } \mathcal{J}$. Conversely, if $\mathbf{w} \in \mathbf{x} + \text{cone } \mathcal{K}$ for some $\mathbf{x} \in \text{cone } \mathcal{J}$, then

$$(5) \quad \mathbf{w} - (\mathbf{x} - \varepsilon_{\mathcal{J}}(\mathbf{x})) \in \varepsilon_{\mathcal{J}}(\mathbf{x}) + \text{cone } \mathcal{K}.$$

Now, $\mathbf{x} - \varepsilon_{\mathcal{J}}(\mathbf{x})$ is a nonnegative multiple of \mathbf{e}_{n+1} , which is in $\text{cone } \mathcal{K}$. Thus, adding $\mathbf{x} - \varepsilon_{\mathcal{J}}(\mathbf{x})$ to both sides of (5) yields $\mathbf{w} \in \varepsilon_{\mathcal{J}}(\mathbf{x}) + \text{cone } \mathcal{K}$. Since $\varepsilon_{\mathcal{J}}(\mathbf{x}) \in \underline{\text{cone}} \mathcal{J}$, this proves the claim. \square

Our ultimate goal is to understand the generating function $\sigma_{\text{cone } \mathcal{J} \oplus \mathcal{K}}$, so we need a version of the disjoint union in Proposition 2.1 that is restricted to the lattice points in $\text{cone}(\mathcal{J} \oplus \mathcal{K})$. This is provided by the following proposition. See also Figure 1.

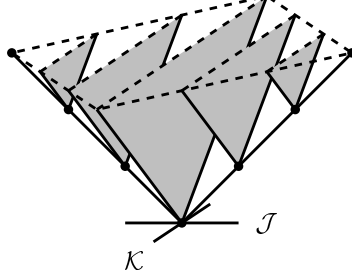


FIGURE 1. A depiction of $\text{cone}(\mathcal{J} \oplus \mathcal{K})$. The dots indicate elements of $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{J}$. The shaded regions represent translations of $\text{cone } \mathcal{K}$ by elements of $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{J}$. The import of Proposition 2.2 is that all lattice points in $\text{cone}(\mathcal{J} \oplus \mathcal{K})$ are within these shaded regions.

Proposition 2.2. *Suppose that $\mathcal{J}, \mathcal{K} \subseteq \mathbb{R}^n$ are convex sets with \mathcal{J} compact such that $\mathcal{J} \oplus \mathcal{K}$ is a free sum. Then*

$$\text{cone}(\mathcal{J} \oplus \mathcal{K})_{\mathbb{Z}} = \bigsqcup_{\mathbf{x} \in \underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{J}} (\mathbf{x} + \text{cone } \mathcal{K})_{\mathbb{Z}}.$$

Proof. The elements of the right-hand side are lattice points that are contained in $\text{cone}(\mathcal{J} \oplus \mathcal{K})$ by the previous proposition. Hence, such elements are in the left-hand side.

To prove the converse containment, let $\mathbf{w} \in \text{cone}(\mathcal{J} \oplus \mathcal{K})_{\mathbb{Z}}$ be given. By Proposition 2.1, there exist $\mathbf{x} \in \underline{\partial} \text{cone } \mathcal{J}$ and $\mathbf{y} \in \text{cone } \mathcal{K}$ such that $\mathbf{w} = \mathbf{x} + \mathbf{y}$. Thus, $\pi(\mathbf{w}) = \pi(\mathbf{x}) + \pi(\mathbf{y})$. Now, $\pi(\mathbf{w})$ is an integer lattice point in $\text{lin}(\mathcal{J} \cup \mathcal{K})$, while $\pi(\mathbf{x}) \in \text{lin } \mathcal{J}$ and $\pi(\mathbf{y}) \in \text{lin } \mathcal{K}$. Since $(\text{lin } \mathcal{J})_{\mathbb{Z}}$ and $(\text{lin } \mathcal{K})_{\mathbb{Z}}$ are complementary sublattices of \mathbb{Z}^n , it follows that $\pi(\mathbf{x}) \in \mathbb{Z}^n$. Therefore, $\mathbf{x} \in \underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{J}$. \square

3. SUFFICIENT CONDITIONS FOR THE MULTIVARIATE BRAUN EQUATION

The multivariate Braun equation (2) does not hold for all free sums $\mathcal{J} \oplus \mathcal{K}$. In this section, we give conditions on \mathcal{J} and \mathcal{K} that suffice to imply equation (2). The conditions we give generalize those originally given by Braun in [5]. In the next section, we will show that, conversely, our conditions are necessary in the case where \mathcal{J} and \mathcal{K} are rational polytopes.

To apply Proposition 2.2, we need to get our hands on the set $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{J}$. The next proposition considers the case where all the elements of this set are integer lattice points.

Proposition 3.1. *Let $\mathcal{J} \subseteq \mathbb{R}^n$ be a compact convex set containing the origin. Then the following conditions are equivalent:*

- (a) $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{J} = (\underline{\partial} \text{cone } \mathcal{J})_{\mathbb{Z}}$,
- (b) $(\underline{\partial} \text{cone } \mathcal{J})_{\mathbb{Z}} = (\text{cone } \mathcal{J})_{\mathbb{Z}} \setminus (\text{cone } \mathcal{J} + \mathbf{e}_{n+1})_{\mathbb{Z}}$,
- (c) $\sigma_{\underline{\partial} \text{cone } \mathcal{J}}(\mathbf{z}) = (1 - z_{n+1}) \sigma_{\text{cone } \mathcal{J}}(\mathbf{z})$.

Proof. We start by proving that (a) and (b) are equivalent. First, note that the set containments

$$\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{J} \supseteq (\underline{\partial} \text{cone } \mathcal{J})_{\mathbb{Z}}$$

and

$$(\underline{\partial} \text{cone } \mathcal{J})_{\mathbb{Z}} \subseteq (\text{cone } \mathcal{J})_{\mathbb{Z}} \setminus (\text{cone } \mathcal{J} + \mathbf{e}_{n+1})_{\mathbb{Z}}$$

always hold. To see that the respective converse containments are equivalent, observe that $\mathbf{x} \mapsto \varepsilon_{\mathcal{J}}(\mathbf{x})$ is a bijection between non-lower-envelope points in $(\text{cone } \mathcal{J})_{\mathbb{Z}} \setminus (\text{cone } \mathcal{J} + \mathbf{e}_{n+1})_{\mathbb{Z}}$ and non-lattice points in $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{J}$, with inverse bijection $(\mathbf{a}, \lambda) \mapsto (\mathbf{a}, \lceil \lambda \rceil)$. Thus, if either containment above is an equality, then so too is the other.

Finally, the left- (resp. right-) hand side of (c) lists the points of the left- (resp. right-) hand side of (b) in generating-function form, so (b) and (c) are equivalent. \square

Theorem 1.2 is now an easy corollary of the previous proposition.

Proof of Theorem 1.2 (stated on p. 3). Since $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{J} = (\underline{\partial} \text{cone } \mathcal{J})_{\mathbb{Z}}$, the set-theoretic equation in Proposition 2.2 can be restated in terms of generating functions as follows:

$$\sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})}(\mathbf{z}) = \sigma_{\underline{\partial} \text{cone } \mathcal{J}}(\mathbf{z}) \sigma_{\text{cone } \mathcal{K}}(\mathbf{z}).$$

The theorem now follows from Proposition 3.1. \square

The conditions in Proposition 3.1 take on an especially nice form when the convex set \mathcal{J} is a rational polytope. We now show that, in this case, these conditions are equivalent to the condition that the polar dual of \mathcal{J} is a lattice polyhedron. We recall the relevant definitions.

The (*polar*) *dual* of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ containing the origin is defined to be the polyhedron

$$\mathcal{P}^{\vee} := \{\varphi \in (\text{lin } \mathcal{P})^* : \varphi(\mathbf{a}) \leq 1 \text{ for all } \mathbf{a} \in \mathcal{P}\},$$

where V^* denotes the set of all real-valued linear functionals on a vector space V . (Note that we use \mathcal{P}^{\vee} to refer to the dual of \mathcal{P} *with respect to the linear span of* \mathcal{P} .) In general, \mathcal{P}^{\vee} may be unbounded, but if $\mathbf{0} \in \mathcal{P}^{\circ}$, then \mathcal{P}^{\vee} is a polytope. Let $\varphi_1, \dots, \varphi_k, \psi_1, \dots, \psi_{\ell} \in (\text{lin } \mathcal{P})^*$ be linear functionals such that

$$\mathcal{P} = \{\mathbf{a} \in \text{lin } \mathcal{P} : \varphi_1(\mathbf{a}), \dots, \varphi_k(\mathbf{a}) \leq 1 \text{ and } \psi_1(\mathbf{a}), \dots, \psi_{\ell}(\mathbf{a}) \leq 0\}.$$

Then \mathcal{P}^{\vee} can be expressed as the Minkowski sum of a polytope and a polyhedral cone in the dual space $(\text{lin } \mathcal{P})^*$ as follows:

$$\mathcal{P}^{\vee} = \text{conv}\{\varphi_1, \dots, \varphi_k\} + \text{pos}\{\psi_1, \dots, \psi_{\ell}\},$$

where $\text{pos } S$ denotes the positive hull $\{\lambda \mathbf{a} : \mathbf{a} \in \text{conv } S \text{ and } \lambda \geq 0\}$ of a set S . We call \mathcal{P}^{\vee} a *lattice polyhedron* if its vertices are in the *dual integer lattice* defined by

$$(\text{lin } \mathcal{P})_{\mathbb{Z}}^* := \{\varphi \in (\text{lin } \mathcal{P})^* : \varphi(\mathbf{a}) \in \mathbb{Z} \text{ for all } \mathbf{a} \in (\text{lin } \mathcal{P})_{\mathbb{Z}}\}.$$

A polytope \mathcal{P} is *reflexive* if both \mathcal{P} and \mathcal{P}^{\vee} are lattice polytopes. The study of reflexive polytopes was initiated by Victor Batyrev, inspired by applications to mirror symmetry in string theory [1].

Hibi [11] showed that a lattice polytope \mathcal{P} containing the origin in its interior is reflexive if and only if $(k\mathcal{P} \setminus (k-1)\mathcal{P})_{\mathbb{Z}} = (\partial(k\mathcal{P}))_{\mathbb{Z}}$ for all integers $k \geq 2$, which, in turn, is equivalent to $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{P} = (\underline{\partial} \text{cone } \mathcal{P})_{\mathbb{Z}}$. Hibi's proofs carry over with virtually no change if we merely assume that \mathcal{P} is rational and contains the origin (not necessarily in its interior). Hibi's arguments then show that \mathcal{P}^{\vee} is a lattice polyhedron if and only if $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{P} = (\underline{\partial} \text{cone } \mathcal{P})_{\mathbb{Z}}$. We include a proof of this equivalence for completeness (Proposition 3.2 below). Non-integral rational polytopes

with lattice duals have appeared, e.g., in [8], which gives a rational analogue of a theorem of Hibi on the Ehrhart series of reflexive polytopes [11].

Proposition 3.2. *Let \mathcal{P} be a rational polytope with $\mathbf{0} \in \mathcal{P}$. Then \mathcal{P}^\vee is a lattice polyhedron if and only if $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{P} = (\underline{\partial} \text{cone } \mathcal{P})_{\mathbb{Z}}$.*

Proof. Suppose that \mathcal{P}^\vee is a lattice polyhedron. It is clear that $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{P} \supseteq (\underline{\partial} \text{cone } \mathcal{P})_{\mathbb{Z}}$. To prove the converse containment, let $\mathbf{x} \in \underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{P}$ be given. By definition of the lower lattice envelope, we have that $\pi(\mathbf{x}) \in \mathbb{Z}^n$. Let $\varphi_1, \varphi_2, \dots, \varphi_k$ be the vertices of \mathcal{P}^\vee , and let $\lambda := \max \{\varphi_1(\pi(\mathbf{x})), \dots, \varphi_k(\pi(\mathbf{x}))\}$. Then $\pi(\mathbf{x}) \in \lambda \mathcal{P}$ while $\pi(\mathbf{x}) \notin (\lambda - \varepsilon) \mathcal{P}$ for $0 < \varepsilon < \lambda$. Thus, $\mathbf{x} = (\pi(\mathbf{x}), \lambda) \in \underline{\partial} \text{cone } \mathcal{P}$. Furthermore, since φ_i is a dual integer lattice point, we have that $\lambda \in \mathbb{Z}$, which implies that $\mathbf{x} \in (\underline{\partial} \text{cone } \mathcal{P})_{\mathbb{Z}}$, proving the desired containment.

Conversely, suppose that \mathcal{P}^\vee has a vertex $\varphi_j \notin (\text{lin } \mathcal{P})_{\mathbb{Z}}^*$. Let $\Lambda \subseteq (\text{lin } \mathcal{P})_{\mathbb{Z}}$ be the sublattice of $(\text{lin } \mathcal{P})_{\mathbb{Z}}$ on which φ_j evaluates as an integer. Thus, Λ is a full-rank proper sublattice of $(\text{lin } \mathcal{P})_{\mathbb{Z}}$. Let F be the facet of \mathcal{P} supported by the hyperplane $\varphi_j = 1$. Then there exists a lattice point $\mathbf{a} \in (\text{pos } F)_{\mathbb{Z}} \setminus \Lambda$. (This may be seen by observing that $\text{pos } F$ is a full-dimensional cone containing some element of Λ in its interior. Hence, $\text{pos } F$ contains some Λ -translate of a fundamental domain of Λ , which in turn contains elements of $\mathbb{Z}^n \setminus \Lambda$.) We then have that $\varphi_j(\mathbf{a}) \notin \mathbb{Z}$ but $(\mathbf{a}, \varphi_j(\mathbf{a})) \in \underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{P}$, so that $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{P} \not\subseteq (\underline{\partial} \text{cone } \mathcal{P})_{\mathbb{Z}}$. \square

As a corollary, we find that the multivariate Braun equation (2) holds when one of the summands is a rational polytope whose polar dual is a lattice polyhedron.

Corollary 3.3. *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a rational polytope such that \mathcal{P}^\vee is a lattice polyhedron, and let $\mathcal{K} \subseteq \mathbb{R}^n$ be a convex set such that $\mathcal{P} \oplus \mathcal{K}$ is a free sum. Then*

$$\sigma_{\text{cone}(\mathcal{P} \oplus \mathcal{K})}(\mathbf{z}) = (1 - z_{n+1}) \sigma_{\text{cone } \mathcal{P}}(\mathbf{z}) \sigma_{\text{cone } \mathcal{K}}(\mathbf{z}).$$

Corollary 3.4. *Let $\mathcal{P}, \mathcal{Q} \subseteq \mathbb{R}^n$ be rational polytopes such that \mathcal{P}^\vee is a lattice polyhedron and $\mathcal{P} \oplus \mathcal{Q}$ is a free sum. Then*

$$\delta_{\mathcal{P} \oplus \mathcal{Q}}(t) = \delta_{\mathcal{P}}(t) \delta_{\mathcal{Q}}(t).$$

In particular, we recover Braun's Theorem 1.1. We remark that Corollary 3.4 also recovers a generalization of Theorem 1.1 due to Braun [5, Corollary 1]. An interesting open question is whether there are lattice polytopes that fit the conditions of Corollary 3.4 but not those of [5, Corollary 1].

4. NECESSARY CONDITIONS FOR THE MULTIVARIATE BRAUN EQUATION

In this section, we prove Theorem 1.3, the converse of Theorem 1.2 in the case where the summands are rational polytopes. That is, we show that, if \mathcal{P} and \mathcal{Q} are rational polytopes containing the origin such that

$$\sigma_{\text{cone}(\mathcal{P} \oplus \mathcal{Q})}(\mathbf{z}) = (1 - z_{n+1}) \sigma_{\text{cone } \mathcal{P}}(\mathbf{z}) \sigma_{\text{cone } \mathcal{Q}}(\mathbf{z}),$$

then either \mathcal{P}^\vee or \mathcal{Q}^\vee is a lattice polyhedron.

Fix a rational polytope $\mathcal{P} \subseteq \mathbb{R}^n$ such that $\mathbf{0} \in \mathcal{P}$, and let $\mathcal{K} \subseteq \mathbb{R}^n$ be a convex set such that $\mathcal{P} \oplus \mathcal{K}$ is a free sum. We begin by considering the rational generating function $\sigma_{\text{cone}(\mathcal{P} \oplus \mathcal{K})}(\mathbf{z})$ in the case where we do not necessarily have $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{P} = (\underline{\partial} \text{cone } \mathcal{P})_{\mathbb{Z}}$. Thus, the results in this section generalize those in Section 3.

Write $d(\mathcal{P})$ for the denominator $\text{den}(\mathcal{P}^\vee)$ of \mathcal{P}^\vee . For each nonnegative integer i , let

$$\begin{aligned}\text{cone}^i \mathcal{P} &:= \text{cone } \mathcal{P} + \frac{i}{d(\mathcal{P})} \mathbf{e}_{n+1}, \\ \text{cone}_i \mathcal{K} &:= \text{cone } \mathcal{K} - \frac{i}{d(\mathcal{P})} \mathbf{e}_{n+1}.\end{aligned}$$

We similarly define the shifted lower envelopes $\underline{\partial} \text{cone}^i \mathcal{P} := \underline{\partial} \text{cone } \mathcal{P} + \frac{i}{d(\mathcal{P})} \mathbf{e}_{n+1}$ and $\underline{\partial} \text{cone}_i \mathcal{K} := \underline{\partial} \text{cone } \mathcal{K} - \frac{i}{d(\mathcal{P})} \mathbf{e}_{n+1}$ of these shifted cones. (Observe that the definition of $\text{cone}_i \mathcal{K}$ depends upon the choice of \mathcal{P} , although this is not reflected in the notation.)

The following result is a generalization of Proposition 3.1 as applied to a rational polytope containing the origin.

Proposition 4.1. *Suppose that $\mathcal{P} \subseteq \mathbb{R}^n$ is a rational polytope with $\mathbf{0} \in \mathcal{P}$, and let $d(\mathcal{P}) := \text{den}(\mathcal{P}^\vee)$. Define the shifted cones $\text{cone}^i \mathcal{P}$ for $0 \leq i \leq d(\mathcal{P})$ as above. Then we have the following:*

- (a) $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{P} = \bigsqcup_{i=0}^{d(\mathcal{P})-1} \left((\underline{\partial} \text{cone}^i \mathcal{P})_{\mathbb{Z}} - \frac{i}{d(\mathcal{P})} \mathbf{e}_{n+1} \right),$
- (b) $(\underline{\partial} \text{cone}^i \mathcal{P})_{\mathbb{Z}} = (\text{cone}^i \mathcal{P})_{\mathbb{Z}} \setminus (\text{cone}^{i+1} \mathcal{P})_{\mathbb{Z}}$ for $0 \leq i \leq d(\mathcal{P}) - 1,$
- (c) $\sigma_{\underline{\partial} \text{cone}^i \mathcal{P}} = \sigma_{\text{cone}^i \mathcal{P}} - \sigma_{\text{cone}^{i+1} \mathcal{P}}$ for $0 \leq i \leq d(\mathcal{P}) - 1.$

Proof. The right-hand side of part (a) is contained in the left-hand side because elements of the right-hand side are points in $\underline{\partial} \text{cone } \mathcal{P}$ that are directly beneath lattice points. To see that the left-hand side of part (a) is contained in the right-hand side, let $\mathbf{x} \in \underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{P}$ be given. It suffices to show that $\mathbf{x} + \frac{i}{d(\mathcal{P})} \mathbf{e}_{n+1} \in \mathbb{Z}^{n+1}$ for some $i \in \{0, \dots, d(\mathcal{P}) - 1\}$. Let

$$\lambda := \max \{ \varphi(\pi(\mathbf{x})) : \varphi \text{ is a vertex of } \mathcal{P}^\vee \},$$

and let $k := \lceil \lambda \rceil$. Thus, $\pi(\mathbf{x}) \in \lambda \mathcal{P}$, but $\pi(\mathbf{x}) \notin (\lambda - \varepsilon) \mathcal{P}$ for all $0 < \varepsilon < \lambda$. Hence, $(\pi(\mathbf{x}), \lambda) \in \underline{\partial} \text{cone } \mathcal{P}$, so $\mathbf{x} = (\pi(\mathbf{x}), \lambda)$. Now, every vertex φ of \mathcal{P}^\vee satisfies $\varphi(\mathbf{a}) \in \frac{1}{d(\mathcal{P})} \mathbb{Z}$ for all $\mathbf{a} \in \mathbb{Z}^n$. Since $\pi(\mathbf{x}) \in \mathbb{Z}^n$, we thus have that $k = \lambda + \frac{i}{d(\mathcal{P})}$ for some $i \in \{0, \dots, d(\mathcal{P}) - 1\}$. Therefore, $\mathbf{x} + \frac{i}{d(\mathcal{P})} \mathbf{e}_{n+1} = (\pi(\mathbf{x}), k) \in \mathbb{Z}^{n+1}$, as required.

To see that the union in part (a) is disjoint, suppose that

$$\mathbf{x}_1 - \frac{i}{d(\mathcal{P})} \mathbf{e}_{n+1} = \mathbf{x}_2 - \frac{j}{d(\mathcal{P})} \mathbf{e}_{n+1}$$

for some $\mathbf{x}_1 \in (\underline{\partial} \text{cone}^i \mathcal{P})_{\mathbb{Z}}$ and $\mathbf{x}_2 \in (\underline{\partial} \text{cone}^j \mathcal{P})_{\mathbb{Z}}$, where, without loss of generality, $0 \leq i \leq j < d(\mathcal{P})$. Then $\mathbf{x}_2 - \mathbf{x}_1 = \left(\frac{j}{d(\mathcal{P})} - \frac{i}{d(\mathcal{P})} \right) \mathbf{e}_{n+1}$ is a lattice point and $0 \leq \frac{j}{d(\mathcal{P})} - \frac{i}{d(\mathcal{P})} < 1$. This implies that $i = j$, showing disjointness and proving part (a).

To prove part (b), suppose that there is an element \mathbf{x} on the right-hand side that is not on the left-hand side. Then, for some integer i such that $0 \leq i \leq d(\mathcal{P}) - 1$ and some λ such that $\frac{i}{d(\mathcal{P})} < \lambda < \frac{i+1}{d(\mathcal{P})}$, we have $\mathbf{x} - \lambda \mathbf{e}_{n+1} \in \underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{P}$. Hence, by part (a), there exist $\mathbf{y} \in \mathbb{Z}^{n+1}$ and $j \in \{0, \dots, d(\mathcal{P}) - 1\}$ such that $\mathbf{y} - \frac{j}{d(\mathcal{P})} \mathbf{e}_{n+1} = \mathbf{x} - \lambda \mathbf{e}_{n+1}$. This implies that $\lambda - \frac{j}{d(\mathcal{P})}$ is an integer, which is a contradiction. This proves part (b). Part (c) follows immediately, since it is a restatement of part (b) in terms of generating functions. \square

Theorem 4.2. *Suppose that $\mathcal{P} \subseteq \mathbb{R}^n$ is a rational polytope and $\mathcal{K} \subseteq \mathbb{R}^n$ is a convex set such that $\mathcal{P} \oplus \mathcal{K}$ is a free sum. Then*

$$\text{cone}(\mathcal{P} \oplus \mathcal{K})_{\mathbb{Z}} = \bigsqcup_{i=0}^{d(\mathcal{P})-1} \bigsqcup_{\mathbf{x} \in (\underline{\partial} \text{cone}^i \mathcal{P})_{\mathbb{Z}}} (\mathbf{x} + \text{cone}_i \mathcal{K})_{\mathbb{Z}}.$$

Therefore,

$$(6) \quad \sigma_{\text{cone}(\mathcal{P} \oplus \mathcal{K})} = \sum_{i=0}^{d(\mathcal{P})-1} (\sigma_{\text{cone}^i \mathcal{P}} - \sigma_{\text{cone}^{i+1} \mathcal{P}}) \sigma_{\text{cone}_i \mathcal{K}}.$$

Proof. By Proposition 2.2,

$$\text{cone}(\mathcal{P} \oplus \mathcal{K})_{\mathbb{Z}} = \bigsqcup_{\mathbf{x} \in \underline{\partial}_{\mathbb{Q}} \text{cone} \mathcal{P}} (\mathbf{x} + \text{cone} \mathcal{K})_{\mathbb{Z}}.$$

By Proposition 4.1, this becomes

$$\begin{aligned} \text{cone}(\mathcal{P} \oplus \mathcal{K})_{\mathbb{Z}} &= \bigsqcup_{i=0}^{d(\mathcal{P})-1} \bigsqcup_{\mathbf{x} \in (\underline{\partial} \text{cone}^i \mathcal{P})_{\mathbb{Z}}} (\mathbf{x} - \frac{i}{d(\mathcal{P})} \mathbf{e}_{n+1} + \text{cone} \mathcal{K})_{\mathbb{Z}} \\ &= \bigsqcup_{i=0}^{d(\mathcal{P})-1} \bigsqcup_{\mathbf{x} \in (\underline{\partial} \text{cone}^i \mathcal{P})_{\mathbb{Z}}} (\mathbf{x} + \text{cone}_i \mathcal{K})_{\mathbb{Z}}. \end{aligned}$$

Hence,

$$\sigma_{\text{cone}(\mathcal{P} \oplus \mathcal{K})} = \sum_{i=0}^{d(\mathcal{P})-1} \sigma_{\underline{\partial} \text{cone}^i \mathcal{P}} \sigma_{\text{cone}_i \mathcal{K}}.$$

Equation (6) now follows from part (c) of Proposition 4.1. \square

Remark 4.3. Some of the terms in Equation (6) may be zero. For example, if \mathcal{P} is the interval $[-2, 3]$, so that $d(\mathcal{P}) = 6$, then $\sigma_{\text{cone}^1 \mathcal{P}} - \sigma_{\text{cone}^2 \mathcal{P}} = 0$. Nonetheless, if $d(\mathcal{P}) > 1$, then $\sigma_{\text{cone}^i \mathcal{P}} - \sigma_{\text{cone}^{i+1} \mathcal{P}} \neq 0$ for some $i \in \{1, \dots, d(\mathcal{P}) - 1\}$ by Proposition 3.2.

Before proving Theorem 1.3, we need two lemmas constraining when lattice points can appear in the shifted lower envelopes of cones over compact convex sets.

Lemma 4.4. *Let $\mathcal{J} \subseteq \mathbb{R}^n$ be a compact convex set, and let ρ be a rational number. Then $(\underline{\partial} \text{cone} \mathcal{J} + \rho \mathbf{e}_{n+1})_{\mathbb{Z}} \neq \emptyset$ if and only if $(\underline{\partial} \text{cone} \mathcal{J} - \rho \mathbf{e}_{n+1})_{\mathbb{Z}} \neq \emptyset$.*

Proof. Since $\rho \in \mathbb{Q}$, a ray R originating at $\rho \mathbf{e}_{n+1}$ contains a lattice point if and only if the inversion of R through $\rho \mathbf{e}_{n+1}$ also contains a lattice point. Hence, the set $\underline{\partial} \text{cone} \mathcal{J} + \rho \mathbf{e}_{n+1}$, which is a union of rays originating at $\rho \mathbf{e}_{n+1}$, contains a lattice point if and only if its inversion through $\rho \mathbf{e}_{n+1}$ contains a lattice point. But $\underline{\partial} \text{cone} \mathcal{J} - \rho \mathbf{e}_{n+1}$ is just the inversion of this latter set through the origin. That is,

$$\underline{\partial} \text{cone} \mathcal{J} - \rho \mathbf{e}_{n+1} = -((\underline{\partial} \text{cone} \mathcal{J} + \rho \mathbf{e}_{n+1}) - \rho \mathbf{e}_{n+1}) + \rho \mathbf{e}_{n+1}.$$

Since inversion through the origin is a lattice-preserving operation, the claim follows. \square

Remark 4.5. The hypothesis in Lemma 4.4 that ρ is rational is necessary. Indeed, if ρ is irrational and $(\partial \text{cone } \mathcal{J} + \rho \mathbf{e}_{n+1})_{\mathbb{Z}} \neq \emptyset$, then we *must* have $(\partial \text{cone } \mathcal{J} - \rho \mathbf{e}_{n+1})_{\mathbb{Z}} = \emptyset$, since otherwise a line containing two lattice points would meet the \mathbf{e}_{n+1} -axis at an irrational point, which is impossible.

Lemma 4.6. *Let \mathcal{Q} be a rational polytope, and let ρ be a real number. If*

$$(\partial \text{cone } \mathcal{Q} + \rho \mathbf{e}_{n+1})_{\mathbb{Z}} \neq \emptyset,$$

then ρ is a rational number.

Proof. Let $\mathbf{x} \in (\partial \text{cone } \mathcal{Q} + \rho \mathbf{e}_{n+1})_{\mathbb{Z}}$, and let F be a facet of cone \mathcal{Q} containing $\mathbf{x} - \rho \mathbf{e}_{n+1}$. Then the supporting hyperplane H of cone \mathcal{Q} at F is a rational hyperplane containing $\rho \mathbf{e}_{n+1} - \mathbf{x}$. Therefore, the translation $H + \mathbf{x}$ by an integer lattice point must meet the \mathbf{e}_{n+1} -axis at a rational point. \square

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3 (stated on p. 3). The “if” direction follows immediately from Corollary 3.3. To prove the converse, suppose that equation (3) holds but that \mathcal{P}^\vee is not a lattice polyhedron. Then, by Proposition 3.2, $\partial_{\mathbb{Z}} \text{cone } \mathcal{P} \neq (\partial \text{cone } \mathcal{P})_{\mathbb{Z}}$. Hence, by Proposition 4.1(a), there exists a maximum integer i with $1 \leq i \leq d(\mathcal{P})-1$ such that $(\partial \text{cone}^i \mathcal{P})_{\mathbb{Z}} \neq \emptyset$. We claim that, in combination with equation (3), this implies that

$$(7) \quad (\text{cone}_i \mathcal{Q})_{\mathbb{Z}} \setminus (\text{cone } \mathcal{Q})_{\mathbb{Z}} = \emptyset.$$

For, suppose otherwise, let $\mathbf{y} \in (\text{cone}_i \mathcal{Q})_{\mathbb{Z}} \setminus (\text{cone } \mathcal{Q})_{\mathbb{Z}}$, and choose $\mathbf{x} \in (\partial \text{cone}^i \mathcal{P})_{\mathbb{Z}}$. On the one hand, by Theorem 4.2, $\mathbf{x} + \mathbf{y} \in \text{cone}(\mathcal{P} \oplus \mathcal{Q})_{\mathbb{Z}}$, or, equivalently, $\mathbf{z}^{\mathbf{x}+\mathbf{y}}$ is a monomial in $\sigma_{\text{cone } \mathcal{P} \oplus \mathcal{Q}}(\mathbf{z})$. On the other hand, since $\mathbf{y} \notin (\text{cone } \mathcal{Q})_{\mathbb{Z}}$, we have

$$\mathbf{x} + \mathbf{y} \notin \bigcup_{i=1}^{d(\mathcal{P})-1} \bigcup_{\mathbf{x} \in (\partial \text{cone}^i \mathcal{P})_{\mathbb{Z}}} (\mathbf{x} + \text{cone } \mathcal{Q})_{\mathbb{Z}}.$$

But, by Proposition 4.1(c), the set on the right-hand side is precisely the set of exponent vectors appearing among the monomials in $(1 - z_{n+1}) \sigma_{\text{cone } \mathcal{P}}(\mathbf{z}) \sigma_{\text{cone } \mathcal{Q}}(\mathbf{z})$, contradicting equation (3).

We now show that $i/d(\mathcal{P}) \geq \frac{1}{2}$. The maximality of i implies that

$$\bigcup_{j=i+1}^{d(\mathcal{P})-1} (\partial \text{cone}^j \mathcal{P})_{\mathbb{Z}} = \emptyset,$$

which, by Lemma 4.4, becomes

$$\bigcup_{j=i+1}^{d(\mathcal{P})-1} (\partial \text{cone}_j \mathcal{P})_{\mathbb{Z}} = \emptyset.$$

Translating by \mathbf{e}_{n+1} then yields

$$\bigcup_{j=i+1}^{d(\mathcal{P})-1} (\partial \text{cone}^{d(\mathcal{P})-j} \mathcal{P})_{\mathbb{Z}} = \bigcup_{j=1}^{d(\mathcal{P})-i-1} (\partial \text{cone}^j \mathcal{P})_{\mathbb{Z}} = \emptyset.$$

Since $(\partial \text{cone}^i \mathcal{P})_{\mathbb{Z}} \neq \emptyset$, we must have $i > d(\mathcal{P}) - i - 1$, or $i/d(\mathcal{P}) \geq \frac{1}{2}$, as claimed.

We now apply similar reasoning to cone \mathcal{Q} . Equation (7) implies that

$$(8) \quad \bigsqcup_{\substack{\rho \in \mathbb{Q}: \\ 0 < \rho \leq i/d(\mathcal{P})}} (\underline{\partial} \text{cone } \mathcal{Q} - \rho \mathbf{e}_{n+1})_{\mathbb{Z}} = \emptyset.$$

Once again applying Lemma 4.4, we get

$$(9) \quad \bigsqcup_{\substack{\rho \in \mathbb{Q}: \\ 0 < \rho \leq i/d(\mathcal{P})}} (\underline{\partial} \text{cone } \mathcal{Q} + \rho \mathbf{e}_{n+1})_{\mathbb{Z}} = \emptyset,$$

while, translating the sets in equation (8) by \mathbf{e}_{n+1} , we have

$$(10) \quad \bigsqcup_{\substack{\rho \in \mathbb{Q}: \\ 0 < \rho \leq i/d(\mathcal{P})}} (\underline{\partial} \text{cone } \mathcal{Q} + (1-\rho)\mathbf{e}_{n+1})_{\mathbb{Z}} = \bigsqcup_{\substack{\rho \in \mathbb{Q}: \\ 1-i/d(\mathcal{P}) \leq \rho < 1}} (\underline{\partial} \text{cone } \mathcal{Q} + \rho \mathbf{e}_{n+1})_{\mathbb{Z}} = \emptyset.$$

Since $i/d(\mathcal{P}) \geq \frac{1}{2}$, we can combine equalities (9) and (10) to conclude that

$$\bigsqcup_{\substack{\rho \in \mathbb{Q}: \\ 0 < \rho < 1}} (\underline{\partial} \text{cone } \mathcal{Q} + \rho \mathbf{e}_{n+1})_{\mathbb{Z}} = \emptyset.$$

Hence, by Lemma 4.6,

$$(\text{cone } \mathcal{Q})_{\mathbb{Z}} \setminus (\text{cone } \mathcal{Q} + \mathbf{e}_{n+1})_{\mathbb{Z}} = (\underline{\partial} \text{cone } \mathcal{Q})_{\mathbb{Z}}.$$

Thus, by Proposition 3.1, we have that $\underline{\partial}_{\mathbb{Z}} \text{cone } \mathcal{Q} = (\underline{\partial} \text{cone } \mathcal{Q})_{\mathbb{Z}}$. Therefore, by Proposition 3.2, \mathcal{Q}^{\vee} is a lattice polyhedron. \square

5. SUMS OF POLYTOPES INTERSECTING AT RATIONAL POINTS

In previous sections, we considered the generating function $\sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})}$ where the join $\mathcal{J} \oplus \mathcal{K}$ was a free sum. In particular, \mathcal{J} and \mathcal{K} intersected only at the origin. Matters are essentially the same if \mathcal{J} and \mathcal{K} intersect at an arbitrary *lattice* point \mathbf{p} in \mathbb{Z}^n , since we can reduce the computation of $\sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})}$ to the previous case via the equation

$$\sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})}(\mathbf{z}) = \sigma_{\text{cone}((\mathcal{J}-\mathbf{p}) \oplus (\mathcal{K}-\mathbf{p}))}(z_1, \dots, z_n, \mathbf{z}^{\alpha(\mathbf{p})}).$$

(Here, in accordance with the convention mentioned in Section 1, $\mathbf{z}^{\alpha(\mathbf{p})}$ denotes the monomial $z_1^{p_1} \cdots z_n^{p_n} z_{n+1}$, where $\mathbf{p} = (p_1, \dots, p_n)$.)

We now turn to the case where \mathcal{J} and \mathcal{K} intersect in an arbitrary rational point in \mathbb{Q}^n . Our results in this section generalize the propositions in Section 2 and some of the results in Section 3. We begin by extending our earlier definitions of lower (lattice) envelopes to accommodate projections that are not in the vertical direction.

Given $\mathbf{p} \in \mathbb{Q}^n$, define $\pi^{\mathbf{p}}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ via $\pi^{\mathbf{p}}(\mathbf{x}) = \pi(\mathbf{x} - x_{n+1}\alpha(\mathbf{p}))$ where x_{n+1} is the last coordinate of \mathbf{x} . Thus, instead of projecting vertically down to \mathbb{R}^n (as in Section 2), $\pi^{\mathbf{p}}$ projects parallel to $\alpha(\mathbf{p})$. Note that $\pi = \pi^{\mathbf{0}}$. However, in general we may not have $\pi^{\mathbf{p}}(\mathbb{Z}^{n+1}) = \mathbb{Z}^n$.

Given a closed linear cone $\mathcal{C} \subseteq \mathbb{R}^{n+1}$ not containing $-\alpha(\mathbf{p})$, define $\varepsilon_{\mathcal{C}}^{\mathbf{p}}: \mathcal{C} \rightarrow \partial \mathcal{C}$ via

$$\varepsilon_{\mathcal{C}}^{\mathbf{p}}(\mathbf{x}) := \mathbf{x} - \max \{ \lambda \in \mathbb{R} : \mathbf{x} - \lambda \alpha(\mathbf{p}) \in \mathcal{C} \} \alpha(\mathbf{p}).$$

Given a compact convex set $\mathcal{J} \subseteq \mathbb{R}^n$, we will write $\varepsilon_{\mathcal{J}}^{\mathbf{p}}$ as an abbreviation for $\varepsilon_{\text{cone } \mathcal{J}}^{\mathbf{p}}$. We then define the **p**-lower envelope $\underline{\partial}^{\mathbf{p}}\mathcal{C}$ of \mathcal{C} via

$$\underline{\partial}^{\mathbf{p}}\mathcal{C} := \varepsilon_{\mathcal{C}}^{\mathbf{p}}(\mathcal{C}).$$

Similar to the lower envelope, the **p**-lower envelope of \mathcal{C} is the set of points in \mathcal{C} that are “minimal in the direction of $\alpha(\mathbf{p})$ ”. Finally, we introduce the notion of **p**-lower lattice envelope of \mathcal{C} , defined as

$$\underline{\partial}_{\mathbb{Z}}^{\mathbf{p}}\mathcal{C} := \varepsilon_{\mathcal{C}}^{\mathbf{p}}(\mathcal{C}_{\mathbb{Z}}).$$

Thus the **p**-lower lattice envelope is the projection of the lattice points in \mathcal{C} in the direction parallel to $\alpha(\mathbf{p})$ onto the **p**-lower envelope of \mathcal{C} . The lower (lattice) envelope of previous sections reappears as the special case $\mathbf{p} = \mathbf{0}$.

We are now ready to state the generalizations of the propositions from Section 2.

Proposition 5.1. *Suppose that $\mathcal{J}, \mathcal{K} \subseteq \mathbb{R}^n$ are convex sets with \mathcal{J} compact. Suppose in addition that the affine spans of \mathcal{J} and \mathcal{K} intersect in exactly one point $\mathbf{p} \in \mathbb{Q}^n$. Then*

$$(11) \quad \text{cone}(\mathcal{J} \oplus \mathcal{K}) = \bigsqcup_{\mathbf{x} \in \underline{\partial}^{\mathbf{p}} \text{cone } \mathcal{J}} (\mathbf{x} + \text{cone } \mathcal{K}).$$

Once we note that, for \mathbf{x} in $\text{cone } \mathcal{J}$, $\pi^{\mathbf{p}}(\mathbf{x})$ is in $\text{lin}(\mathcal{J} - \mathbf{p})$, the proof of this proposition is the same as the proof of Proposition 2.1 with the appropriate replacements (such as π replaced by $\pi^{\mathbf{p}}$, and $\varepsilon_{\mathcal{J}}$ replaced by $\varepsilon_{\mathcal{J}}^{\mathbf{p}}$).

We now seek a restriction of equation (11) to lattice points that is in the spirit of Proposition 2.2. To this end, we define an analogue of free sums that we call *affine free sums*. Recall that, for the join $\mathcal{J} \oplus \mathcal{K}$ to be a free sum, we required that $(\text{lin } \mathcal{J})_{\mathbb{Z}}$ and $(\text{lin } \mathcal{K})_{\mathbb{Z}}$ be complementary sublattices of \mathbb{Z}^n . One complication of our present case is that $\pi^{\mathbf{p}}(\mathbf{x})$ is not necessarily a lattice point for every lattice point \mathbf{x} in $\text{cone } \mathcal{J}$. Thus, we consider the refinement $\Lambda^{\mathbf{p}} := \pi^{\mathbf{p}}(\mathbb{Z}^{n+1})$ of \mathbb{Z}^n . There are several equivalent characterizations of this lattice:

- (1) $\Lambda^{\mathbf{p}} = \pi^{\mathbf{p}}(\mathbb{Z}^{n+1})$.
- (2) $\Lambda^{\mathbf{p}}$ is the lattice in \mathbb{R}^n generated by $\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{p}$ under integer linear combinations.
- (3) $\Lambda^{\mathbf{p}} = \bigsqcup_{k=0}^{r-1} (\mathbb{Z}^n - k\mathbf{p})$, where $r := \text{den}(\mathbf{p})$ is the least common multiple of the denominators of the coordinates of \mathbf{p} .

We adapt our earlier notation and terminology to work with the lattice $\Lambda^{\mathbf{p}}$ as follows. For a subset S of \mathbb{R}^n , let $S_{\Lambda^{\mathbf{p}}}$ denote the set of points in $S \cap \Lambda^{\mathbf{p}}$. We say that two sublattices $\mathcal{L}, \mathcal{M} \subseteq \Lambda^{\mathbf{p}}$ are *complementary sublattices of $\Lambda^{\mathbf{p}}$* if each lattice point in $(\text{lin}(\mathcal{L} \cup \mathcal{M}))_{\Lambda^{\mathbf{p}}}$ can be written uniquely as a sum of lattice points in \mathcal{L} and \mathcal{M} .

We call a join $\mathcal{J} \oplus \mathcal{K}$ of convex sets in \mathbb{R}^n an *affine free sum* when \mathcal{J} and \mathcal{K} intersect at a point $\mathbf{p} \in \mathbb{Q}^n$ such that $(\text{lin}(\mathcal{J} - \mathbf{p}))_{\Lambda^{\mathbf{p}}}$ and $(\text{lin}(\mathcal{K} - \mathbf{p}))_{\Lambda^{\mathbf{p}}}$ are complementary sublattices of $\Lambda^{\mathbf{p}}$. Equivalently, $\mathcal{J} \oplus \mathcal{K}$ is an affine free sum if every integer lattice point in the linear span of the cone over $\mathcal{J} \oplus \mathcal{K}$ is a sum (non-uniquely) of lattice points in the respective linear spans of the cones over \mathcal{J} and over \mathcal{K} . That is, $\mathcal{J} \oplus \mathcal{K}$ is an affine free sum if

$$\text{lin}(\alpha(\mathcal{J} \cup \mathcal{K}))_{\mathbb{Z}} = \text{lin}(\alpha(\mathcal{J}))_{\mathbb{Z}} + \text{lin}(\alpha(\mathcal{K}))_{\mathbb{Z}}.$$

Proposition 5.2. *Suppose that $\mathcal{J}, \mathcal{K} \subseteq \mathbb{R}^n$ are convex sets with \mathcal{J} compact such that $\mathcal{J} \oplus \mathcal{K}$ is an affine free sum of convex sets intersecting at $\mathbf{p} \in \mathbb{Q}^n$. Then*

$$\text{cone}(\mathcal{J} \oplus \mathcal{K})_{\mathbb{Z}} = \bigsqcup_{\mathbf{x} \in \partial_{\mathbb{Z}}^{\mathbf{p}} \text{cone } \mathcal{J}} (\mathbf{x} + \text{cone } \mathcal{K})_{\mathbb{Z}}.$$

Proof. Elements on the right-hand side are integer lattice points that are contained in $\text{cone}(\mathcal{J} \oplus \mathcal{K})$ by Proposition 5.1. Hence, such elements are in the left-hand side.

To prove the converse containment, let $\mathbf{w} \in \text{cone}(\mathcal{J} \oplus \mathcal{K})_{\mathbb{Z}}$ be given. Then by the previous proposition, $\mathbf{w} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x} \in \partial^{\mathbf{p}} \text{cone } \mathcal{J}$ and $\mathbf{y} \in \text{cone } \mathcal{K}$. Thus, $\pi^{\mathbf{p}}(\mathbf{w}) = \pi^{\mathbf{p}}(\mathbf{x}) + \pi^{\mathbf{p}}(\mathbf{y})$. Now, $\pi^{\mathbf{p}}(\mathbf{w})$ is in $\text{lin}((\mathcal{J} \cup \mathcal{K}) - \mathbf{p})_{\Lambda^{\mathbf{p}}}$, while $\pi^{\mathbf{p}}(\mathbf{x}) \in \text{lin}(\mathcal{J} - \mathbf{p})$ and $\pi^{\mathbf{p}}(\mathbf{y}) \in \text{lin}(\mathcal{K} - \mathbf{p})$. Thus, the complementarity of $(\text{lin}(\mathcal{J} - \mathbf{p}))_{\Lambda^{\mathbf{p}}}$ and $(\text{lin}(\mathcal{K} - \mathbf{p}))_{\Lambda^{\mathbf{p}}}$ implies that $\pi^{\mathbf{p}}(\mathbf{x})$ is in $\Lambda^{\mathbf{p}}$. Hence there exists a non-negative integer λ such that $(\pi^{\mathbf{p}}(\mathbf{x}), 0) + \lambda\alpha(\mathbf{p}) = (\pi^{\mathbf{p}}(\mathbf{x}) + \lambda\mathbf{p}, \lambda)$ is an integer lattice point in $\text{cone } \mathcal{J}$. Since $\varepsilon_{\mathcal{J}}^{\mathbf{p}}((\pi^{\mathbf{p}}(\mathbf{x}) + \lambda\mathbf{p}, \lambda)) = \mathbf{x}$, we have $\mathbf{x} \in \varepsilon_{\mathcal{J}}^{\mathbf{p}}((\text{cone } \mathcal{J})_{\mathbb{Z}})$, and the result follows. \square

We now turn to the rational generating function $\sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})}(\mathbf{z})$ and state the generalizations of Proposition 3.1 and Theorem 1.2.

Proposition 5.3. *Fix a compact convex set $\mathcal{J} \subseteq \mathbb{R}^n$ containing $\mathbf{p} \in \mathbb{Q}^n$. Let $r := \text{den}(\mathbf{p})$. Then the following are equivalent:*

- (a) $\partial_{\mathbb{Z}}^{\mathbf{p}} \text{cone } \mathcal{J} = (\partial^{\mathbf{p}} \text{cone } \mathcal{J})_{\mathbb{Z}}$,
- (b) $(\partial^{\mathbf{p}} \text{cone } \mathcal{J})_{\mathbb{Z}} = (\text{cone } \mathcal{J})_{\mathbb{Z}} \setminus (\text{cone } \mathcal{J} + r\alpha(\mathbf{p}))_{\mathbb{Z}}$,
- (c) $\sigma_{\partial^{\mathbf{p}} \text{cone } \mathcal{J}}(\mathbf{z}) = (1 - \mathbf{z}^{r\alpha(\mathbf{p})}) \sigma_{\text{cone } \mathcal{J}}(\mathbf{z})$.

Proof. We first show that (a) and (b) are equivalent. By definition of the \mathbf{p} -lower envelope and \mathbf{p} -lower lattice envelope, we have $(\partial^{\mathbf{p}} \text{cone } \mathcal{J})_{\mathbb{Z}} \subseteq \partial_{\mathbb{Z}}^{\mathbf{p}} \text{cone } \mathcal{J}$ and $(\partial^{\mathbf{p}} \text{cone } \mathcal{J})_{\mathbb{Z}} \subseteq (\text{cone } \mathcal{J})_{\mathbb{Z}} \setminus (\text{cone } \mathcal{J} + r\alpha(\mathbf{p}))_{\mathbb{Z}}$. As in the proof of Proposition 3.1, we observe that $\mathbf{x} \mapsto \varepsilon_{\mathcal{J}}^{\mathbf{p}}(\mathbf{x})$ is a bijection between non-lower-envelope points in $(\text{cone } \mathcal{J})_{\mathbb{Z}} \setminus (\text{cone } \mathcal{J} + r\alpha(\mathbf{p}))_{\mathbb{Z}}$ and non-lattice points in $\partial_{\mathbb{Z}}^{\mathbf{p}} \text{cone } \mathcal{J}$, with the inverse given by $\mathbf{x} \mapsto \mathbf{x} + \min\{\lambda \in \mathbb{R} : \mathbf{x} + \lambda\alpha(\mathbf{p}) \in (\text{cone } \mathcal{J})_{\mathbb{Z}}\} \alpha(\mathbf{p})$. The rest of the proof is the same as the proof of Proposition 3.1. \square

Theorem 5.4. *Suppose that $\mathcal{J}, \mathcal{K} \subseteq \mathbb{R}^n$ are convex sets with \mathcal{J} compact such that $\mathcal{J} \oplus \mathcal{K}$ is an affine free sum of convex sets intersecting at $\mathbf{p} \in \mathbb{Q}^n$. Further suppose that $\partial_{\mathbb{Z}}^{\mathbf{p}} \text{cone } \mathcal{J} = (\partial^{\mathbf{p}} \text{cone } \mathcal{J})_{\mathbb{Z}}$. Then*

$$\sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})}(\mathbf{z}) = \left(1 - \mathbf{z}^{r\alpha(\mathbf{p})}\right) \sigma_{\text{cone } \mathcal{J}}(\mathbf{z}) \sigma_{\text{cone } \mathcal{K}}(\mathbf{z}),$$

where $r := \text{den}(\mathbf{p})$.

The proof is the same as the proof of Theorem 1.2 with the appropriate replacements.

Example 5.5. Let \mathcal{J} be the line segment from $(0,0)$ to $(1,0)$ in \mathbb{R}^2 and let \mathcal{K} be the line segment from $(\frac{1}{2}, -1)$ to $(\frac{1}{2}, 1)$ in \mathbb{R}^2 . Then \mathcal{J} and \mathcal{K} intersect at $\mathbf{p} := (\frac{1}{2}, 0)$. The join $\mathcal{J} \oplus \mathcal{K}$ is an affine free sum. The \mathbf{p} -lower envelope of $\text{cone } \mathcal{J}$ is the boundary of the cone, and the set of lattice points in the boundary is precisely the set $(\text{cone } \mathcal{J})_{\mathbb{Z}} \setminus (\text{cone } \mathcal{J} + r\alpha(\mathbf{p}))_{\mathbb{Z}}$, with $\alpha(\mathbf{p}) = (\frac{1}{2}, 0, 1)$ and $r := \text{den}(\mathbf{p}) = 2$.

Thus, \mathcal{J} satisfies the conditions in Proposition 5.3. Hence, Theorem 5.4 applies, yielding

$$\begin{aligned} \sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})}(z_1, z_2, z_3) &= (1 - z_1 z_3^2) \sigma_{\text{cone} \mathcal{J}}(z_1, z_2, z_3) \sigma_{\text{cone} \mathcal{K}}(z_1, z_2, z_3) \\ &= (1 - z_1 z_3^2) \frac{1}{(1 - z_3)(1 - z_1 z_3)} \frac{1 + z_1 z_2^{-1} z_3^2 + z_1 z_3^2 + z_1 z_2 z_3^2}{(1 - z_1 z_2^{-2} z_3^2)(1 - z_1 z_2^2 z_3^2)}. \end{aligned}$$

Example 5.6. Theorem 5.4 need not hold if we drop the condition that $\partial_{\mathbb{Z}}^{\mathbf{p}} \text{cone } \mathcal{J} = (\partial^{\mathbf{p}} \text{cone } \mathcal{J})_{\mathbb{Z}}$. If we keep \mathcal{J} the same set as in Example 5.5, but let \mathcal{K} be the line segment from $(\frac{1}{3}, -1)$ to $(\frac{1}{3}, 1)$ in \mathbb{R}^2 , then $\mathbf{p} = (\frac{1}{3}, 0)$, $\alpha(\mathbf{p}) = (\frac{1}{3}, 0, 1)$ and $r = 3$. The \mathbf{p} -lower envelope of $\text{cone } \mathcal{J}$ is still the boundary of the cone, but there are now lattice points in the set $(\text{cone } \mathcal{J})_{\mathbb{Z}} \setminus (\text{cone } \mathcal{J} + r\alpha(\mathbf{p}))_{\mathbb{Z}}$ that are not in the boundary of the cone. Thus, the conditions in Proposition 5.3 are not true of \mathcal{J} , and so we would need to use generalizations of results from Section 4 to compute $\sigma_{\text{cone}(\mathcal{J} \oplus \mathcal{K})}(\mathbf{z})$. We do not develop such generalizations here.

We now consider an important class of polytopes for which the conditions in Proposition 5.3 are true, so that Theorem 5.4 applies when one of the summands is a polytope from this class. A lattice polytope \mathcal{P} is *Gorenstein of index k* if there exists a lattice point \mathbf{m} such that $k\mathcal{P} - \mathbf{m}$ is a reflexive polytope. In particular, \mathbf{m} is the unique lattice point in $k\mathcal{P}^{\circ}$. (Here we write S° for the interior of a set S relative to the subspace topology on $\text{lin } S$.) The recent paper [13] discusses Braun's formula in the context of Gorenstein polytopes and nef-partitions.

Proposition 5.7. *Suppose that $\mathcal{P} \subseteq \mathbb{R}^n$ is a Gorenstein polytope of index k . Let \mathbf{m} be the unique lattice point in $k\mathcal{P}^{\circ}$, and let $\mathbf{p} := \frac{1}{k}\mathbf{m}$. Then $\partial_{\mathbb{Z}}^{\mathbf{p}} \text{cone } \mathcal{P} = (\partial^{\mathbf{p}} \text{cone } \mathcal{P})_{\mathbb{Z}}$.*

Proof. Since $\mathbf{p} \in \mathcal{P}^{\circ}$, we have that $\partial \text{cone } \mathcal{P} = \partial \text{cone } \mathcal{P}$. It is well known that the Gorenstein property implies that $(\text{cone } \mathcal{P})_{\mathbb{Z}}^{\circ} = (\text{cone } \mathcal{P} + r\alpha(\mathbf{p}))_{\mathbb{Z}}$, where $r := \text{den}(\mathbf{p})$ (see, e.g., [6]). The result follows. \square

Corollary 5.8. *Suppose that $\mathcal{P} \subseteq \mathbb{R}^n$ is a Gorenstein polytope of index k . Let \mathbf{m} be the unique lattice point in $k\mathcal{P}^{\circ}$, and let $\mathbf{p} := \frac{1}{k}\mathbf{m}$. Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a convex set containing \mathbf{p} such that $\mathcal{J} \oplus \mathcal{K}$ is an affine free sum. Then*

$$\sigma_{\text{cone}(\mathcal{P} \oplus \mathcal{K})}(\mathbf{z}) = \left(1 - \mathbf{z}^{k\alpha(\mathbf{p})}\right) \sigma_{\text{cone } \mathcal{P}}(\mathbf{z}) \sigma_{\text{cone } \mathcal{K}}(\mathbf{z}).$$

Example 5.5 is an instance of this corollary, as the line segment \mathcal{J} in that example is a Gorenstein polytope of index 2 with $\mathbf{m} = (1, 0)$.

In Section 3, we noted that the conditions in Proposition 3.1 applied to reflexive polytopes. Indeed, in that context, the integrality of vertices was unimportant; all that we needed was that the polar dual be a lattice polyhedron. It is natural to expect that the Gorenstein condition in Proposition 5.7 can similarly be replaced with some weaker condition that admits non-lattice polytopes. For example, one might hope that, in Theorem 5.7, we could take \mathcal{P} to be any rational polytope such that, for some integer k and some lattice point $\mathbf{m} \in k\mathcal{P}$, $(k\mathcal{P} - \mathbf{m})^{\vee}$ is a lattice polyhedron. Unfortunately, this is not the case in general, as the following example shows.

Example 5.9. Let $\mathcal{P} := [\frac{1}{4}, \frac{3}{4}] \subseteq \mathbb{R}^1$. Observe that $2\mathcal{P} = [\frac{1}{2}, \frac{3}{2}]$ contains the lattice point $\mathbf{m} := 1$ and that the polar dual of $2\mathcal{P} - \mathbf{m} = [-\frac{1}{2}, \frac{1}{2}]$ is the lattice

polytope $[-2, 2] \subseteq (\mathbb{R}^1)^*$. Nonetheless, putting $\mathbf{p} := \frac{1}{2}\mathbf{m}$, the \mathbf{p} -lower lattice envelope of cone \mathcal{P} contains the non-lattice point $(\frac{1}{2}, 2)$. Therefore, the conclusion of Theorem 5.7 is not true of \mathcal{P} .

REFERENCES

- [1] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom. **3** (1994), no. 3, 493–535, [arXiv:alg-geom/9310003](#).
- [2] M. Beck and S. Hoşten, *Cyclotomic polytopes and growth series of cyclotomic lattices*, Math. Res. Lett. **13** (2006), no. 4, 607–622.
- [3] M. Beck and S. Robins, *Computing the continuous discretely: Integer-point enumeration in polyhedra*, Undergraduate Texts in Mathematics, Springer, New York, 2007, Electronically available at <http://math.sfsu.edu/beck/ccd.html>.
- [4] G. M. Bergman, *On lattices of convex sets in \mathbb{R}^K* , Algebra Universalis **53** (2005), no. 2-3, 357–395.
- [5] B. Braun, *An Ehrhart series formula for reflexive polytopes*, Electron. J. Combin. **13** (2006), no. 1, Note 15, 5 pp. (electronic).
- [6] W. Bruns and T. Römer, *h -vectors of Gorenstein polytopes*, J. Combin. Theory Ser. A **114** (2007), no. 1, 65–76.
- [7] E. Ehrhart, *Sur les polyèdres rationnels homothétiques à n dimensions*, C. R. Acad. Sci. Paris **254** (1962), 616–618.
- [8] M. H. J. Fiset and A. M. Kasprzyk, *A note on palindromic δ -vectors for certain rational polytopes*, Electron. J. Combin. **15** (2008), no. 1, Note 18, 4.
- [9] M. Henk, J. Richter-Gebert, and G. M. Ziegler, *Basic properties of convex polytopes*, Handbook of discrete and computational geometry, CRC Press Ser. Discrete Math. Appl., CRC, Boca Raton, FL, 1997, pp. 243–270.
- [10] T. Hibi, *Algebraic Combinatorics on Convex Polytopes*, Carlsaw Publications, 1992.
- [11] ———, *Dual polytopes of rational convex polytopes*, Combinatorica **12** (1992), no. 2, 237–240.
- [12] P. McMullen, *Constructions for projectively unique polytopes*, Discrete Math. **14** (1976), no. 4, 347–358.
- [13] B. Nill and J. Schepers, *Gorenstein polytopes and their stringy e -functions*, preprint ([arXiv:1005.5158](#)), 2010.
- [14] M. A. Perles and G. C. Shephard, *Facets and nonfacets of convex polytopes*, Acta Math. **119** (1967), 113–145.
- [15] R. P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997.

(Matthias Beck) DEPARTMENT OF MATHEMATICS, SAN FRANCISCO STATE UNIVERSITY, SAN FRANCISCO, CA 94132, USA

E-mail address: mattbeck@sfsu.edu

(Pallavi Jayawant) DEPARTMENT OF MATHEMATICS, BATES COLLEGE, LEWISTON, ME 04240, USA

E-mail address: pjayawan@bates.edu

(Tyrrell B. McAllister) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WYOMING, LARAMIE, WY 82071, USA

E-mail address: tmcallis@uwyo.edu